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Classical r -matrices and the Feigin–Odesskii algebra via Hamiltonian and Poisson reductions

H W Braden¹, V A Dolgushev^{2,3}, M A Olshanetsky³ and A V Zotov³

¹ School of Mathematics, University of Edinburgh, UK

² Massachusetts Institute of Technology, Cambridge, MA, USA

³ Institute for Theoretical and Experimental Physics, Moscow, Russia

E-mail: hwb@ed.ac.uk, vald@mit.edu, olshanet@heron.itep.ru and zotov@gate.itep.ru

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Abstract

We present a formula for a classical r -matrix of an integrable system obtained by Hamiltonian reduction of some free field theories using pure gauge symmetries. The framework of the reduction is restricted only by the assumption that the respective gauge transformations are Lie group ones. Our formula is in terms of Dirac brackets, and some new observations on these brackets are made. We apply our method to derive a classical r -matrix for the elliptic Calogero–Moser system with spin starting from the Higgs bundle over an elliptic curve with marked points. In the paper, we also derive a classical Feigin–Odesskii algebra by a Poisson reduction of some modification of the Higgs bundle over an elliptic curve. This allows us to include integrable lattice models in a Hitchin-type construction.

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1. Introduction

A classical r -matrix structure is an important tool for investigating integrable systems [1–3]. It encodes the Hamiltonian structure of the Lax equation, provides the involution of integrals of motion and gives a natural framework for quantizing integrable systems in a quantum group theoretic setting [4].

The aim of this paper is severalfold. First, we present a formula for the classical r -matrices of integrable systems, derived in the framework of Dirac’s Hamiltonian reduction for systems involving only gauge transformations. In the process we shall derive new results describing the Dirac brackets. As an application of our general formula we shall calculate a classical r -matrix for the elliptic Calogero–Moser system with spin. Following the works [5–7] we start from the Higgs bundle over an elliptic curve with a marked point. This is a free theory with a trivial r -matrix. After the Dirac procedure we come to the desired r -matrix. Finally, we derive the

classical Feigin–Odesskii algebra [8] in a similar fashion. Here the corresponding unreduced space is similar to the cotangent bundle of the centrally extended loop group $\hat{L}(GL_N)$. The Poisson structure on this space is a particular example of a general construction proposed in [9]. In this way we derive the corresponding quadratic Poisson algebra using the Hitchin approach [10]. Thus we have managed to include integrable lattice models in the general Hitchin construction.

Our work is motivated by [11, 12] where the authors use a gauge invariant extension of Lax matrices to derive classical r -matrices for the Toda chain, and for the trigonometric and elliptic (spin zero) Calogero–Moser systems. In [11], the Hamiltonian reduction is considered on the cotangent bundle to a finite-dimensional Lie group, while in the second paper the construction is generalized to the case of the central extension of the loop group. In this context [13] also warrants mention. There the authors consider a special case of Poisson reduction on Poisson–Lie groupoids in order to obtain new examples of the class of dynamical r -matrices defined by Etingof and Varchenko [14].

In our paper we propose a very general framework for Hamiltonian reduction by pure gauge symmetries that allows one to derive a classical r -matrix. This framework is based on rather general assumptions: essentially that the first-class constraints generate the adjoint action of some Lie algebra on a (Lie algebra valued) function on the unreduced space, which in turn reduces to the Lax equation, and that the Poisson brackets between these elements are already cast into r -matrix form.

The organization of the paper is as follows. In the second section, we outline the general framework of the Hamiltonian reduction which allows us to derive r -matrices for integrable systems. In the third section, we discuss the Dirac brackets and the r -matrices more generally, relating each of these to generalized inverses. This sheds some new light on Dirac’s brackets. The fourth section consists of our first example, where we apply our method to derive the classical r -matrix structure for the elliptic Calogero–Moser system with spin [5, 7] using its Čech-like Hitchin description [6, 10]. The fifth section is devoted to the derivation of the classical Feigin–Odesskii algebra [8] from some free field theory. This theory is a modification of the Higgs bundle related to the holomorphic GL_N -bundle of degree one over an elliptic curve. Instead of a Higgs field that takes values in the endomorphisms of the bundle we consider the field of automorphisms of the bundle. It is a Poisson manifold [9] and we show that the elliptic Belavin–Drinfeld r -matrix [15] is naturally obtained in the context of such a reduction. In the concluding section we discuss a possible generalization of our reduction technique to Hitchin systems on curves of higher genus. An appendix is devoted to the special elliptic functions we use.

Throughout the paper we use standard notation from quantum group theory and integrable systems to express the Poisson brackets between entries of matrices. Thus, for example, the r -matrix equation cast as

$$\{L \otimes L\} = [r, L \otimes 1] - [r^T, 1 \otimes L] \quad (1)$$

becomes

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]. \quad (2)$$

If the Lax matrix takes values in some Lie algebra \mathfrak{g} (or representation of this), the r -matrix takes values in $\mathfrak{g} \otimes \mathfrak{g}$ or its corresponding representation. We may expand quantities in terms of a basis of \mathfrak{g} as follows. Let t_μ denote such a basis with $[t_\mu, t_\nu] = c_{\mu\nu}^\lambda t_\lambda$ defining the structure constants of \mathfrak{g} . Suppose $\phi(t_\mu) = X_\mu$, where ϕ yields the desired representation of the

Lie algebra; we may take this to be a faithful representation. With $L = \sum_{\mu} L^{\mu} X_{\mu}$ the left-hand side of (1) becomes

$$\{L \otimes L\} = \sum_{\mu, \nu} \{L^{\mu}, L^{\nu}\} X_{\mu} \otimes X_{\nu} \tag{3}$$

while upon setting $r = r^{\mu\nu} X_{\mu} \otimes X_{\nu}$ and $r^T = r^{\nu\mu} X_{\mu} \otimes X_{\nu}$ the right-hand side yields

$$\begin{aligned} [r, L \otimes 1] - [r^T, 1 \otimes L] &= r^{\mu\nu} L^{\lambda} ([X_{\mu}, X_{\lambda}] \otimes X_{\nu} - X_{\nu} \otimes [X_{\mu}, X_{\lambda}]) \\ &= (r^{\tau\nu} c_{\tau\lambda}^{\mu} L^{\lambda} - r^{\tau\mu} c_{\tau\lambda}^{\nu} L^{\lambda}) X_{\mu} \otimes X_{\nu}. \end{aligned} \tag{4}$$

Concretely, in the standard basis of gl_N

$$(e_{ij})_{mn} = \delta_{im} \delta_{jn}$$

we have $\{X_{\mu}\} = \{e_{ij}\}$ and (2) takes the form

$$\{L_{ij}, L_{kl}\} = \sum_m (r_{imkl} L_{mj} - L_{im} r_{mjkl} - r_{kmij} L_{ml} + L_{km} r_{mlij}) \tag{5}$$

where

$$r = \sum_{i,j,k,l} r_{ijkl} e_{ij} \otimes e_{kl}. \tag{6}$$

Then r_{21} denotes the function with permuted indices $r_{21}(z) = r_{klij}(z) e_{ij} \otimes e_{kl}$.

At the outset let us record that under a gauge transformation r -matrices transform as

$$\begin{aligned} r^G(z, z') &= \Lambda_1^{-1} \Lambda_2^{-1} r(z, z') \Lambda_1 \Lambda_2 + \Lambda_1^{-1} \Lambda_2^{-1} \{l_2(z'), \Lambda_1\} \Lambda_2 \\ &\quad + \frac{1}{2} [\Lambda_1^{-1} \Lambda_2^{-1} \{\Lambda_1, \Lambda_2\}, \Lambda_2^{-1} l_2(z') \Lambda_2]. \end{aligned} \tag{7}$$

Also, given a Lax operator, r -matrices are far from being uniquely defined. We will specify this ambiguity in due course.

2. The formal r -matrix via Hamiltonian reduction

In this section, we show that if a classical integrable system is obtained by a Hamiltonian reduction involving purely gauge symmetries, and so corresponding first-class constraints, then the system admits a canonical r -matrix. The word ‘formal’ in the title of the section stresses the fact that in many interesting cases the reduction is performed in the context of a field theory, and consequently for such systems the formula for the r -matrix must be properly defined.

Our approach makes three quite general assumptions regarding the Hamiltonian reduction that allows us to calculate the r -matrix. These are

1. There exists an element Φ , which upon reduction becomes the Lax matrix, and this takes values in some Lie algebra \mathfrak{g} .
2. The gauge transformations of $\Phi \in \mathfrak{g}$ can be represented as

$$\Phi \mapsto h^{-1} \Phi h \tag{8}$$

where h is assumed to take values in the Lie group G corresponding to the Lie algebra \mathfrak{g} .

3. The Poisson brackets between the ‘entries’ of the element Φ in the unreduced space have been cast into the r -matrix form before reduction.

The first two assumptions are quite natural if we wish to obtain an integrable system together with its Lax representation. The second assumption implies that the Poisson brackets between the first-class constraints T_a generating the gauge symmetries, and Φ can be written in terms of the commutators

$$\{T_a, \Phi\} = [e_a, \Phi] \quad (9)$$

where $\{e_a\}$ is a basis of the Lie algebra of gauge transformations (which may be a subalgebra of \mathfrak{g}). The third assumption is more elaborate. It means that there exists a classical r -matrix $r^0 \in \mathfrak{g} \otimes \mathfrak{g}$, defining the Poisson brackets in the total or unreduced space between the ‘entries’ of Φ

$$\{\Phi_1, \Phi_2\} = [r_{12}^0, \Phi_1] - [r_{21}^0, \Phi_2]. \quad (10)$$

Our first calculation below shows how the initial r -matrix r^0 produces, under reduction, the desired classical r -matrix of the integrable system. One may feel that we have simply transferred the problem of constructing an r -matrix for a reduced system to one of constructing an r -matrix for an unreduced system. In practice constructing the latter r -matrix is often easier, but we will deal with the construction of r^0 in the next section.

Together with some gauge fixing conditions $\chi^a = 0$ the first-class constraints form a system of the second-class constraints $\{\sigma_\alpha\} = \{T_a, \chi^b\}$ [16]. Using these constraints we may define the Dirac bracket, which allows us to calculate the reduced Poisson brackets in terms of the Poisson brackets on the unreduced phase space. Thus, in order to calculate the Poisson bracket between two observables f, k of the reduced phase space, we have to first calculate the Dirac bracket between arbitrary continuations F, K of f and k in the unreduced space:

$$\{F, K\}_{\text{DB}} = \{F, K\} - \{F, \sigma_\alpha\} C^{\alpha\beta} \{\sigma_\beta, K\}. \quad (11)$$

Finally, the reduced Poisson bracket between f and k is obtained by restricting expression (11) to the surface of the second-class constraints $\{\sigma_\alpha = 0\}$. Here $\|C^{\alpha\beta}\|$ is the matrix inverse to $\|C_{\alpha\beta}\| = \{\sigma_\alpha, \sigma_\beta\}$ and $\{, \}$ is the Poisson bracket on the unreduced space.

Since under reduction the element $\Phi \in \mathfrak{g}$ becomes the Lax matrix, the Poisson brackets between the ‘entries’ of the Lax matrix are just the on-shell Dirac brackets between the respective ‘coordinates’ Φ on \mathfrak{g} . Our third assumption automatically casts the first term in the respective Dirac bracket (11) into the required form (10). Perhaps the surprising thing is that the remaining terms of the on-shell Dirac bracket (11) may also be cast into r -matrix form as a consequence of the second assumption. To see this we observe that (on-shell) C takes the block form

$$C = \begin{pmatrix} 0 & \{T_a, \chi^b\} \\ \{\chi^a, T_b\} & \{\chi^a, \chi^b\} \end{pmatrix} \equiv \begin{pmatrix} 0 & P \\ -P^T & \{\chi^a, \chi^b\} \end{pmatrix} \quad P_a^b = \{T_a, \chi^b\} \quad (12)$$

using the fact that $\{T_a, T_b\} = 0$ as they are first-class constraints. Then C^{-1} has the form

$$C^{-1} = \begin{pmatrix} P^{T-1} \{\chi^a, \chi^b\} P^{-1} & -P^{T-1} \\ P^{-1} & 0 \end{pmatrix} \equiv \begin{pmatrix} Q & -P^{T-1} \\ P^{-1} & 0 \end{pmatrix} \quad (13)$$

where

$$Q^{ab} = (P^{-1})_c^a \{\chi^c, \chi^d\} (P^{-1})_d^b. \quad (14)$$

Thus the only non-vanishing on-shell entries of the matrix $\|C^{\alpha\beta}\|$ are

$$C^{\chi^a T_b} = -C^{T_b \chi^a} = (P^{-1})_a^b \quad \text{together with} \quad C^{T_a T_b} = Q^{ab}.$$

Using these the final term in $\{\Phi_1, \Phi_2\}_{\text{DB}}$ may be written in the r -matrix form as follows:

$$\begin{aligned} -\{\Phi_1, \sigma_\alpha\} C^{\alpha\beta} \{\sigma_\beta, \Phi_2\} &= -\{\Phi_1, \chi^a\} (P^{-1})^b_a \{T_b, \Phi_2\} + \{\Phi_1, T_a\} (P^{-1})^a_b \{\chi^b, \Phi_2\} \\ &\quad - \{\Phi_1, T_a\} Q^{ab} \{T_b, \Phi_2\} \\ &= -[(P^{-1})^a_b e_a \otimes \{\chi^b, \Phi\}, \Phi_1] + [\{\chi^a, \Phi\} (P^{-1})^b_a \otimes e_b, \Phi_2] \\ &\quad + \frac{1}{2} [e_a \otimes Q^{ab} \{T_b, \Phi\}, \Phi_1] - \frac{1}{2} [\{\Phi, T_a\} Q^{ab} \otimes e_b, \Phi_2]. \end{aligned}$$

Here we have made use of equation (9). Upon combining all the terms (10) for the on-shell Dirac bracket $\{\Phi_1, \Phi_2\}_{\text{DB}}$ we obtain the desired classical r -matrix.

Theorem 1. *Under the three assumptions given we have*

$$r = (r^0 - (P^{-1})^a_b e_a \otimes \{\chi^b, \Phi\} + \frac{1}{2} Q^{ab} e_a \otimes \{T_b, \Phi\})|_{\text{on-shell}} \tag{15}$$

where P is given in (12) and Q in (14).

The terms modifying r^0 have the same general form as the infinitesimal form of (7). We shall further explain this remark later in the paper.

In section 4, we apply the method outlined here to derive the classical r -matrix for the elliptic Calogero–Moser system with spin. Before doing so, however, let us return to the question of r^0 , and place our construction in a somewhat broader context.

3. Dirac brackets, generalized inverses and r -matrices

Our assumptions have *a priori* assumed the existence of an r -matrix r^0 for the unreduced system. In many examples such unreduced r -matrices are easy to construct: when the unreduced matrices Φ depend only on half of the phase space variables, such as the momenta, then for example $r^0 = 0$ is a possible solution. This however is not always the case, and in this section we shall discuss the issue of the existence of r^0 . This will enable us both to place our Dirac Bracket construction in a wider context and to elaborate on the remarks of [17] concerning this reduction procedure. For concreteness we shall phrase our discussion in the language of finite-dimensional matrices (as we did in the previous section).

The construction of r -matrices is essentially an algebraic operation. This algebraic nature was clarified in [18] where a necessary and sufficient condition was given for the existence of an r -matrix based upon the fundamental Poisson brackets of the Lax matrix (here Φ). The novel part of this investigation was the use of generalized inverses (and the construction thereof for generic elements of the adjoint representation of a Lie algebra \mathfrak{g}). We shall recall these notions in our present setting, relating them to Dirac brackets. In so doing we will arrive at some new results pertaining to Dirac brackets.

We begin with generalized inverses. Let A be an arbitrary matrix. (In particular A need neither be square nor invertible.) A matrix G_A is said to be a generalized inverse of A provided

$$AG_A A = A. \tag{16}$$

Such a matrix always exists, yet need not be unique. We can further require

$$G_A A G_A = G_A. \tag{17}$$

Again such a generalized inverse always exists, yet need not be unique. Observe that given a G_A satisfying (16) and (17) we have at hand projection operators $P_1 = G_A A$ and $P_2 = A G_A$ which satisfy

$$A P_1 = P_2 A = A \quad P_1 G_A = G_A P_2 = G_A. \tag{18}$$

One may specify a unique generalized inverse (which always exists), the Moore–Penrose inverse, by additionally requiring

$$(AG_A)^\dagger = AG_A \quad (G_AA)^\dagger = G_AA. \tag{19}$$

Let us remark that the adjoint $(\)^\dagger$ here is defined with respect to a given inner product, the Moore–Penrose inverse satisfying a norm-minimizing condition. Typically, this is an Hermitian inner product and the adjoint is the Hermitian conjugate. We shall denote by A^+ the Moore–Penrose inverse of A . (Accounts of generalized inverses may be found in [19–22].) Geometrically, the Moore–Penrose inverse may be constructed by orthogonally projecting onto the subspace on which A has maximal rank and inverting the resulting matrix. We also record an alternative characterization [23]. Denote by \tilde{A} and \tilde{G} the matrices

$$\tilde{A} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \quad \tilde{G} = \begin{pmatrix} 0 & 0 \\ G_A & 0 \end{pmatrix}.$$

Then

1. (16) and (17) $\iff \langle \tilde{A}, \tilde{G}, [\tilde{A}, \tilde{G}] \rangle$ form an sl_2 -triple.
2. Further $G_A = A^+ \iff [\tilde{A}, \tilde{G}]$ is Hermitian.

Let us now relate Dirac brackets to generalized inverses. We may suppose our (say unreduced) phase space has canonical Poisson brackets

$$\{x^i, p_j\} = \delta_j^i \tag{20}$$

so that

$$\{F, K\} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial p} \right) J \begin{pmatrix} \frac{\partial K}{\partial x} \\ \frac{\partial K}{\partial p} \end{pmatrix} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{21}$$

(We shall give a coordinate-independent description in due course.) In terms of the second-class constraints σ_α we consider the matrices

$$a_\alpha^k = \frac{\partial \sigma_\alpha}{\partial p_k} \quad b_{\alpha k} = \frac{\partial \sigma_\alpha}{\partial x^k} \quad C_{\alpha\beta} = \{\sigma_\alpha, \sigma_\beta\}. \tag{22}$$

In general, we have a different number of constraints c from the number of coordinates, and the matrices a and b are not invertible. Given that the Moore–Penrose inverse is constructed by an orthogonal projection, and that Dirac’s brackets give us a projection onto the constraint surface, the following may not be surprising (though it should be more widely known).

Theorem 2. *With definitions (22) let $A = (b, a)$ and set $G_A = \begin{pmatrix} a^T C^{-1} \\ -b^T C^{-1} \end{pmatrix}$. Then*

(i)

$$AG_AA = A \quad G_AAG_A = G_A \quad \text{and} \quad AG_A = \text{Id}_{c \times c} = (AG_A)^\dagger.$$

(ii) *If $P_1 = G_AA$ then $P_1J = JP_1^T$.*

(iii)

$$\{F, K\}_{\text{DB}} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial p} \right) (1 - P_1) J \begin{pmatrix} \frac{\partial K}{\partial x} \\ \frac{\partial K}{\partial p} \end{pmatrix}.$$

(iv) *P_1 is self-adjoint with respect to any inner product of the form*

$$\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle = \mathbf{u}^T P_1^T Q P_1 \mathbf{v} + \mathbf{u}^T (1 - P_1^T) Q (1 - P_1) \mathbf{v}$$

where Q is an invertible symmetric matrix. For such an inner product $G_A = A^+$ is the Moore–Penrose inverse of A .

Proof. The proof of these statements is straightforward. Observe that from the basic Poisson bracket

$$C = ba^T - ab^T \tag{23}$$

we have that

$$(b, a) \begin{pmatrix} a^T C^{-1} \\ -b^T C^{-1} \end{pmatrix} = (ba^T - ab^T)C^{-1} = 1_{c \times c}.$$

Thus $P_2 = AG_A = P_2^\dagger$ (for any choice of the adjoint). It is then clear that $G_A = \begin{pmatrix} a^T C^{-1} \\ -b^T C^{-1} \end{pmatrix}$ satisfies (16) and (17). Direct calculation shows that

$$P_1 = G_A A = \begin{pmatrix} a^T C^{-1} b & a^T C^{-1} a \\ -b^T C^{-1} b & -b^T C^{-1} a \end{pmatrix}$$

whence $P_1 J = J P_1^T$, and (iii) follows from (11). We will establish the theorem upon showing that P_1 is self-adjoint. It is now that the issue of inner product confronts us. We shall discuss this more generally after proving the final assertion of the theorem. Now using the specified inner product and that P_1, P_1^T are projectors we have that

$$\langle\langle P_1 \mathbf{u}, \mathbf{v} \rangle\rangle = \mathbf{u}^T P_1^T \cdot P_1^T Q P_1 \mathbf{v} = \mathbf{u}^T P_1^T Q P_1 \mathbf{v} = \mathbf{u}^T P_1^T Q P_1 \cdot P_1 \mathbf{v} = \langle\langle \mathbf{u}, P_1 \mathbf{v} \rangle\rangle.$$

Then upon comparing with the definition of the adjoint, $\langle\langle P_1 \mathbf{u}, \mathbf{v} \rangle\rangle = \langle\langle \mathbf{u}, P_1^\dagger \mathbf{v} \rangle\rangle$, we see that $P_1 = P_1^\dagger$ and so is self-adjoint. Therefore, $G_A = A^+$ is the Moore–Penrose inverse for such an inner product. \square

Let us now describe this theorem in a more geometric manner. Denote the unreduced phase space by (P, ω) . Given a function F on P we have a corresponding vector field X_F given via

$$dF = i_{X_F} \omega = \omega(X_F, \cdot).$$

The Poisson bracket of two functions is then

$$\{F, K\} = \omega(X_F, X_K) = dF(X_K).$$

Thus with $\omega = \sum dq^j \wedge dp_j$ we have

$$X_F = \frac{\partial F}{\partial p_j} \frac{\partial}{\partial x^j} - \frac{\partial F}{\partial x^j} \frac{\partial}{\partial p_j}$$

and the Poisson brackets (21). Our constraints describe a reduced phase space $V = \cap_{\alpha=1}^c \{\sigma_\alpha = 0\} \subset P$. Important for us is that V is not an arbitrary submanifold but a symplectic submanifold, and the inclusion $i : V \hookrightarrow P$ gives the symplectic form $\omega_V = i^* \omega$ on V . Because ω_V is nondegenerate, $T_z V \cap (T_z V)^\omega = 0$, where $(T_z V)^\omega = \{u \in T_z P \mid \omega(u, v) = 0 \forall v \in V\}$. Then

$$T_z P = T_z V \oplus (T_z V)^\omega$$

and we have a projection $\pi : TP \rightarrow TV$. One finds that

$$\pi(X_F) = X_F - \{F, \sigma_\alpha\} C^{\alpha\beta} X_{\sigma_\beta} = X_F - X_{\sigma_\beta} C^{\beta\alpha} d\sigma_\alpha(X_F) = [1 - X_{\sigma_\beta} C^{\beta\alpha} d\sigma_\alpha](X_F)$$

and the Dirac brackets are

$$\begin{aligned} \{F, K\}_{\text{DB}} &= \{F|_V, K|_V\} = \omega(\pi(X_F), \pi(X_K)) = dF(\pi(X_K)) \\ &= dF([1 - X_{\sigma_\beta} C^{\beta\alpha} d\sigma_\alpha](X_K)). \end{aligned}$$

(Observe that $\pi^2 = \pi$ follows from $d\sigma_\alpha(X_{\sigma_\beta}) = C_{\alpha\beta}$. Such a projection operator will exist even for a Poisson manifold P provided $C_{\alpha\beta}$ is invertible: in this case we have

$T_z P = \ker \pi \oplus \text{Im } \pi$ and $\text{Im } \pi$ is again Poisson.) Comparison with the corresponding coordinate expression given by (iii) of the theorem then shows that

$$\pi = 1 - P_1.$$

The projection operator P_1 is then $P_1 : TP \rightarrow (T_z V)^\omega$, which is spanned by the vector fields $\{X_{\sigma_a}\}$. Everything thus far has taken place within the realm of symplectic geometry: P_1 is a symplectic projector and is self-adjoint with respect to the nondegenerate bilinear form ω ,

$$\omega(P_1(X_F), X_K) = \omega(P_1(X_F), P_1(X_K)) = \omega(X_F, P_1(X_K)) = \omega(X_F, P_1^\dagger(X_K)).$$

(This is the content of (ii) of the theorem with $\omega(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T J \mathbf{v}$.) To talk of orthogonal projection we need an inner product. While the kinetic energy of a natural Hamiltonian system can provide this an inner product is an additional ingredient. However, as (iv) of the theorem shows, there are a large class of inner products for which π becomes an orthogonal projection and we have

$$T_z P = T_z V \oplus (T_z V)^\perp. \quad (24)$$

In practice one often further restricts attention to compatible inner products for which we have

$$\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle = \omega(J\mathbf{u}, \mathbf{v}) \quad \omega(\mathbf{u}, \mathbf{v}) = \omega(J\mathbf{u}, J\mathbf{v}) \quad J^2 = -\text{Id}$$

and respecting the vector space decomposition (24).

We remark that generalized inverses have been discussed in various connections with singular Lagrangian systems. Broadbridge and Petersen [24] refine an observation of Duffin [25] that a generalized inverse may be used to go from a singular Lagrangian system, such as those arising when constraints are implemented by Lagrange multipliers, and a corresponding Hamiltonian system. The reduced Hamiltonians appearing correspond to those of de Leeuw *et al* [26]. (These works place various restrictions on the nature of the constraints.) Because the Moore–Penrose inverse may be calculated very efficiently their use in solving constrained dynamical systems is important. Kalaba and Udawadia formulate a large class of constrained Lagrangian systems as a quadratic programming problem, and use the Moore–Penrose inverse to solve these [27].

Finally, just as the Dirac bracket can be expressed in terms of generalized inverses, so too can the solution of the r -matrix equation (2) or (10). From (4) the r -matrix equation takes the explicit form

$$b = a^T r - r^T a$$

where we have set $a^{\mu\nu} = c_{\mu\lambda}^\nu L^\lambda \equiv -ad(L)_\mu^\nu$ and $b^{\mu\nu} = \{L^\mu, L^\nu\}$. The solutions of this equation have been studied [28].

Theorem 3. *Let g be a generalized inverse of $a = -ad(L)$ satisfying (16) and (17). Then, with $P_1 = ga$ and $P_2 = ag$, the r -matrix equation (1) has solutions if and only if*

$$(1 - P_1^T)b(1 - P_1) = 0 \quad (25)$$

in which case the general solution is

$$r = \frac{1}{2}g^T b P_1 + g^T b(1 - P_1) + (1 - P_2^T)Y + (P_2^T Z P_2)a \quad (26)$$

where Y is arbitrary and Z is only constrained by the requirement that $P_2^T Z P_2$ be symmetric.

The theorem then constructs the r -matrix in terms of a generalized inverse to adL and describes the ambiguity of the solution. Although the general solution appears to depend on the generalized inverse, the work cited shows that changing the generalized inverse only changes the solution by such terms. Further, the generalized inverse of a generic adL may

be constructed. Thus the existence of an r -matrix has been reduced to the single consistency equation (25) and the construction of a generalized inverse to $ad(L)$.

Having now addressed the issue of r^0 , and indeed the general construction of r -matrices, we will return to the particular construction given by the Dirac reduction procedure of the previous section. We shall illustrate this in turn by two examples.

4. Example 1: derivation of the r -matrix for the elliptic Calogero–Moser system with spin

In this section, we show that a classical r -matrix for the elliptic Calogero–Moser system with spin can easily be derived with the help of Dirac brackets. One surprise (for us) is that in calculating the r -matrix we do not make use of the identities for elliptic functions that typically underlie these integrable systems.

In deriving the r -matrix for the elliptic Calogero–Moser system with spin we will divide the Hamiltonian reduction procedure into two stages. In the first stage we omit gauge transformations related to the parabolic subgroup. This yields the formula for the classical r -matrix of the spin Calogero–Moser system with the maximal spin sector.

After completing the first stage of the reduction we are left with residual action of the parabolic subgroup on a finite-dimensional phase space. Here it is possible to derive the desired classical r -matrix using any concrete fixing of the residual gauge symmetry. An alternative method also exists, making use of a gauge invariant extension of the Lax matrix. We will adopt the latter approach to show that upon further reduction the r -matrix obtained in the first stage becomes a classical r -matrix for the elliptic spin Calogero–Moser system.

The Hamiltonian reduction that leads to the elliptic Calogero–Moser system with spin was originally presented in [5], where the phase space of the system was realized as the cotangent bundle to the moduli space of topologically trivial holomorphic bundles (the Higgs bundles) over the torus with a marked point. We are going to employ this construction in deriving the respective classical r -matrix.

For our purposes we mostly follow the paper [6] and use a Čech-like description of the moduli space. We will realize the torus Σ_τ as a quotient of $\mathbb{C}/q\mathbb{Z}$ with $q = e^{2\pi i\tau}$, $\text{Im } \tau > 0$. We choose as the fundamental domain the annulus $\text{Ann}_\tau = \{|q^{1/2}| < |z| < |q^{-1/2}|\}$.

We define a holomorphic vector bundle E_N over Σ_τ by the matrix transition function $g(z) \in GL_N(\mathbb{C})$ that is holomorphic in a neighbourhood of the contour $\gamma = \{|z| = |q^{1/2}|\}$. In this way $g(z)$ represents a Čech cocycle. The set of these fields we denote as $\mathcal{L} = \{g\}$.

Let \mathcal{G} be the gauge group of holomorphic maps $f(z)$ of the annulus Ann_τ to $GL_N(\mathbb{C})$ such that

$$f(z)|_{z=1} = I \tag{27}$$

where I is the identity matrix. It acts on \mathcal{L} as

$$g(z) \mapsto f(z)g(z)f^{-1}(q^{-1}z). \tag{28}$$

Observe that the Lie algebra $\text{Lie}(\mathcal{G})$ consists of matrix-valued holomorphic functions $\varepsilon(z) : \text{Ann}_\tau \mapsto gl_N(\mathbb{C})$ vanishing at the point $z = 1$

$$\varepsilon(z)|_{z=1} = 0.$$

The quotient space under the gauge action (28)

$$\mathcal{M} = \mathcal{L}/\mathcal{G} \tag{29}$$

is the moduli space of the holomorphic bundles over Σ_τ with the marked point $z = 1$. Note that upon reduction with respect to adjoint action of a parabolic subgroup $\mathcal{P} \subset SL_N(\mathbb{C})$ the space

(29) becomes the moduli space of the holomorphic bundles over Σ_τ with a quasi-parabolic structure at $z = 1$.

The cotangent bundle $T^*\mathcal{L}$ is called the Higgs bundle. The dual field (the Higgs field) is a one-form

$$\eta = \eta(z) \frac{dz}{z}$$

taking values in $\text{End}(E_N) = \text{Lie algebra } gl_N$. The field $\eta(z)$ is holomorphic in a neighbourhood of γ_1 .

The bundle $T^*\mathcal{L}$ is the non-reduced phase space with the symplectic form

$$\Omega = \frac{1}{2\pi i} \oint_{\gamma_1} \text{Tr} \delta(g^{-1}(z)\eta) \wedge \delta g(z). \tag{30}$$

We lift the gauge group action (28) on $T^*\mathcal{L}$ as

$$\eta(z) \mapsto f(z)\eta(z)f^{-1}(z). \tag{31}$$

The symplectic form (30) is invariant with respect to this gauge action. In this way it produces the moment map

$$\mu : T^*\mathcal{L} \mapsto \text{Lie}^*(\mathcal{G}) \tag{32}$$

which is defined as

$$\mu[\eta, g](\varepsilon) = \oint_{\gamma_1} \text{Tr}(g^{-1}(z)\eta g(z)\varepsilon(q^{-1}z) - \eta\varepsilon(z)) \quad \varepsilon \in \text{Lie}(\mathcal{G}). \tag{33}$$

The zero level of the moment map (33) is the surface in $T^*\mathcal{L}$ where $\eta(z)$ can be extended meromorphically from the boundary (γ_1, γ_2) inside the annulus with at most single pole of the first order at the point $z = 1$ and such that for $|z| = |q^{1/2}|$,

$$\eta(q^{-1}z) = g^{-1}(z)\eta(z)g(z). \tag{34}$$

While a generic topologically trivial bundle over a torus can be decomposed into a direct sum of line bundles [29], a generic holomorphic bundle with a marked point can differ from a direct sum of line bundles by the conjugation of a constant matrix $\alpha \in SL_N(\mathbb{C})$. In other words, using the gauge group action (28) one can transform a generic field $g(z)$ to a constant matrix of the following form:

$$h = \alpha \begin{pmatrix} e^{-2\pi i u_1} & 0 & \dots & 0 \\ 0 & e^{-2\pi i u_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{-2\pi i u_N} \end{pmatrix} \alpha^{-1}. \tag{35}$$

In particular, this means that

$$\partial_z g_{ii}(z) = 0 \tag{36}$$

is a proper gauge condition for our system.

It is easy to see that the entries of the matrix α are defined by h up to the scaling transformations

$$\alpha_{ij} \mapsto \alpha_{ij} \lambda_j \quad \lambda_j \neq 0. \tag{37}$$

while u_i and $u'_i = u_i + n_1 + n_2\tau, n_1, n_2 \in \mathbb{Z}$ gives the same holomorphic bundle over Σ_τ . Thus, an open dense set of the moduli space \mathcal{M} (29) can be parametrized by the points of the set⁴

$$(J \times \mathbb{C}P^{N-1})^N \tag{38}$$

⁴ In fact we slightly abuse the parametrization omitting the quotient with respect to the symmetric group of permutations S_N , which plays an analogous role as the Weyl group for the rational Calogero–Moser system.

where J is the Jacobian of the curve Σ_τ . In the general context of holomorphic vector bundles over Riemann surfaces this is known as the Tyurin parametrization [30]. This parametrization has found recent application in a general description of Hitchin systems and their r -matrices [31–33].

Expanding the functions $g(z)$ and $\eta(z)$ in a neighbourhood of the contour γ ,

$$\eta_{ij}(z) = \sum_{a \in \mathbb{Z}} \eta_{ij}^a z^a \quad g_{ij}(z) = \sum_{a \in \mathbb{Z}} g_{ij}^a z^a \tag{39}$$

we readily find the respective Poisson brackets to be

$$\{\eta_{ij}^a, \eta_{kl}^b\} = -\eta_{il}^{a+b} \delta_{kj} + \eta_{kj}^{a+b} \delta_{il} \quad \{g_{ij}^a, \eta_{kl}^b\} = g_{kj}^{a+b} \delta_{il} \quad \{g_{ij}^a, g_{kl}^b\} = 0. \tag{40}$$

In these terms the expansion coefficients of the constraint $\mu[\eta, g] = 0$ (33) are of the form

$$M_{ij}^a = \sum_{b,c,k,l} (g^{-1})_{ik}^{b-c} \eta_{kl}^c g_{lj}^{a-b} q^a - \sum_{b,c,k,l} (g^{-1})_{ik}^{b-c} \eta_{kl}^c g_{lj}^{-b} - \eta_{ij}^a + \eta_{ij}^0 \quad a \neq 0 \tag{41}$$

and the gauge fixing conditions (36) can be rewritten as follows:

$$g_{ij}^a = 0 \quad a \neq 0. \tag{42}$$

Our second-class constraints are then

$$M_{ij}^a = 0 \quad (a \neq 0) \quad g_{ij}^a = 0 \quad (a \neq 0). \tag{43}$$

The field $g(z)$ being restricted to the constraint surface (43) becomes the constant matrix $g(z) = h$ (35) while the field η yields the Lax matrix of the elliptic Calogero–Moser system with maximal spin sector,

$$\eta(z) = l(z) = \alpha \tilde{l}(z) \alpha^{-1} \tag{44}$$

with

$$\tilde{l}_{ii}(z) = -\frac{v_i}{2\pi i} \quad \tilde{l}_{ij}(z) = -\sum_{k=1}^N \beta_{ik} \alpha_{kj} \phi(z, u_{ij}) \quad i \neq j \quad u_{ij} = u_i - u_j.$$

Here u_i, α_{ij}, v_j and β_{ij} are coordinates on the reduced phase space $T^*\mathcal{M} \approx T^*(J \times \mathbb{C}P^{N-1})^N$. Because α_{ij} (for each fixed $j = 1, \dots, N$) are homogeneous coordinates on the respective projective spaces $\mathbb{C}P^{N-1}$, the symplectic form on $T^*\mathcal{M}$ is obtained from the form

$$\Omega_1 = \sum_{i=1}^N dv_i \wedge du_i + \sum_{i,j=1}^N d\beta_{ij} \wedge d\alpha_{ji} \tag{45}$$

by symplectic reduction on the first-class constraint surface (arising from the generators of (37))

$$\sum_{k=1}^N \beta_{ik} \alpha_{ki} = 0. \tag{46}$$

The function $\phi(z, u)$ in (44) can be represented in the following form:

$$\phi(z, u) = \sum_{a \in \mathbb{Z}} \frac{z^a}{q^a e^{2\pi i u} - 1} \quad |q| < |z| < 1 \quad u \neq a + b\tau \quad a, b \in \mathbb{Z}. \tag{47}$$

We note that the Lax matrix of the Hitchin system with a quasi-parabolic structure at the marked point $z = 1$ is obtained from (44) by reduction with respect to the adjoint action of the corresponding parabolic subgroup $\mathcal{P} \subset SL_N(\mathbb{C})$.

In order to construct the r -matrix using our theorem we must determine the Dirac bracket. First we calculate the matrix of the Poisson brackets between the constraints (43) and then determine the inverse matrix. The non-vanishing on-shell brackets are found to be

$$\{g_{ij}^a, M_{kl}^b\} = \delta_{a+b,0}(q^{-a}h_{il}\delta_{kj} - h_{kj}\delta_{il}). \tag{48}$$

To present the inverse matrix we adopt the following convention for the entries of the inverse matrix: $(C^{Mg})_{ab}^{ijkl}$ stands for the entry with M_{ij}^a being the first index and g_{kl}^b being the second. Then the non-vanishing entries of the inverse matrix can be written in the following form:

$$(C^{Mg})_{ab}^{ijkl} = -(C^{gM})_{ba}^{kl ij} = \sum_{m,n=1}^N \frac{\delta_{a+b,0}\alpha_{lm}(\alpha^{-1})_{mi}\alpha_{jn}(\alpha^{-1})_{nk}}{q^a e^{-2\pi i u_n} - e^{-2\pi i u_m}}. \tag{49}$$

In deriving the r -matrix via the Dirac bracket we will exploit the following simple property:

$$\{M_{ij}^a, \eta^b\} = [e_{ji}, \eta^{a+b}] - [e_{ji}, \eta^b] \tag{50}$$

which is just the counterpart of the general relation (9).

Now the expression for the Dirac bracket (11) between the expansion coefficients of the field $\eta(z)$ consists of two parts. The first part is just the initial Poisson bracket in the extended phase space. This may be rewritten as

$$\{\eta_1^a, \eta_2^b\} = \frac{1}{2} \sum_{i,j} [e_{ij} \otimes e_{ji}, \eta_1^{a+b}] - \frac{1}{2} \sum_{i,j} [e_{ij} \otimes e_{ji}, \eta_2^{a+b}]. \tag{51}$$

The second part reflects the reduction. With $\{\eta_1^a, \eta_2^b\}_{DB} = \{\eta_1^a, \eta_2^b\} + D^{ab}$ the on-shell expression for the second part is

$$D^{ab} = \sum_{m,n=1}^N \frac{\alpha_{im}(\alpha^{-1})_{ml}\alpha_{kn}(\alpha^{-1})_{nj}}{q^b e^{-2\pi i u_{mn}} - 1} ([e_{ij} \otimes e_{kl}, \eta_1^{a+b}] - [e_{ij} \otimes e_{kl}, \eta_1^a]) - \sum_{m,n=1}^N \frac{\alpha_{im}(\alpha^{-1})_{ml}\alpha_{kn}(\alpha^{-1})_{nj}}{q^a e^{-2\pi i u_{mn}} - 1} ([e_{kl} \otimes e_{ij}, \eta_2^{b+a}] - [e_{kl} \otimes e_{ij}, \eta_2^b]) \tag{52}$$

where $u_{mn} = u_m - u_n$.

Finally, to calculate the Dirac bracket between the fields $\eta(z)$ and $\eta(z')$ we must perform the sum

$$\{\eta_1(z), \eta_2(z')\}_{DB} = \sum_{a,b \in \mathbb{Z}} \{\eta_1^a, \eta_2^b\}_{DB} z^a (z')^b. \tag{53}$$

Such sums require some care because separately certain series in the expression for the Dirac bracket (51), (52) do not converge. These must be accurately combined in order to get a finite answer.

For example, if we choose $|q| < |z'| < |z| < 1$, hence $|q| < |z'/z| < 1$ and $1 < |z/z'| < |q^{-1}|$, then the second sum in (52) gives a divergent series in (53). However, if we add to the above terms expression (51) we obtain a convergent series and the desired r -matrix for the Lax matrix (44) takes the following form:

$$r(z, z') = \alpha_1 \alpha_2 \left[(E(z'/z) - E(z')) \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} (\phi(z'/z, u_{ji}) - \phi(z', u_{ji})) e_{ij} \otimes e_{ji} \right] \alpha_1^{-1} \alpha_2^{-1} \tag{54}$$

where

$$\alpha_1 = \alpha \otimes I \quad \alpha_2 = I \otimes \alpha$$

and the function $E(z)$ can be represented in the form of the following series:

$$E(z) = \sum_{a \in \mathbb{Z}, a \neq 0} \frac{z^a}{q^a - 1} - 1 \quad |q| < |z| < 1. \tag{55}$$

Remark 1. The classical r -matrices for the spin Calogero–Moser systems were originally found in the paper [34]. In contrast to [34] where the final formulae, both for the Lax matrix and for the r -matrix, were defined on the elliptic curve only after an additional auxiliary reduction, our formulae (44) and (54) are defined on the elliptic curve from the outset.

Remark 2. The elliptic Calogero–Moser system with maximal spin sector can also be described in terms of a Lax matrix and r -matrix which are functions, rather than sections of some bundles on the elliptic curve. Namely, the Lax matrix (44) and the r -matrix (54) are gauge equivalent to the following matrix-valued functions on Σ_τ

$$l_{ij}^K(w) = \sum_{k,l=1}^N \alpha_{ik} B_{kl}(w) (\alpha^{-1})_{lj} \quad B_{ii} = -\frac{v_i}{2\pi i} \tag{56}$$

$$B_{ij}(w) = -\sum_{k=1}^N \beta_{ik} \alpha_{kj} \frac{\theta_{11}(w - u_j) \theta_{11}(w + u_j - u_i) \theta_{11}(u_i) \theta'_{11}(0)}{\theta_{11}(w) \theta_{11}(w - u_i) \theta_{11}(u_i - u_j) \theta_{11}(u_j)} \quad i \neq j$$

$$\begin{aligned} \tilde{r}(w, w') &= \sum_{i,j=1}^N (E(w - w') + E(w')) e_{ij} \otimes e_{ji} \\ &\quad - \sum_{i,j,k,l=1}^N \alpha_{il} (\alpha^{-1})_{lk} (E(w - u_l) + E(u_l)) e_{ij} \otimes e_{jk} \end{aligned} \tag{57}$$

where

$$w = \frac{1}{2\pi i} \ln z \quad w' = \frac{1}{2\pi i} \ln z'$$

and

$$\theta_{11}(w) = \sum_{m \in \mathbb{Z}} \exp(\pi i \tau (m + 1/2)^2 + 2\pi i (m + 1/2)(w + 1/2)).$$

The Lax matrix for the spin Calogero–Moser system in the form (56) was originally obtained in the paper by Krichever [32], while formula (57) for the corresponding r -matrix has been given recently in [33].

Remark 3. In obtaining result (54) we have only used the definition of the Dirac bracket and *not* the elliptic function identities which are unavoidable for checking that (54) is indeed the r -matrix for the Lax matrix (44). Of course the functions our procedure yields as series are, as shown in the appendix, expressible as elliptic functions.

The resulting r -matrix (54) is universal in a sense that it allows us to derive a classical r -matrix for the respective Hitchin system on the elliptic curve with an arbitrary quasi-parabolic structure at the marked point. We will conclude this section by elaborating upon this point.

Let \mathcal{P} be a parabolic subgroup of $SL_N(\mathbb{C})$. Then the Hitchin system on the elliptic curve with the corresponding quasi-parabolic structure at the marked point is obtained from the

maximal spin system described above upon the further Hamiltonian reduction with respect to the following action of the parabolic subgroup \mathcal{P} ,

$$\alpha_{ij} \mapsto \sum_{k=1}^N q_{ik} \alpha_{kj} \quad \alpha_{ij} \mapsto \sum_{k=1}^N \beta_{ik} (q^{-1})_{kj} \quad u_i \mapsto u_i \quad v_i \mapsto v_i. \quad (58)$$

Here $q = (q_{ij}) \in \mathcal{P}$. The momentum map of the action (58) is found to be

$$\mu^P : T^* \mathcal{M} \mapsto \text{Lie}^*(\mathcal{P}) \quad \mu^P[\alpha, \beta](X) = \text{Tr}(\alpha\beta X) \quad X \in \text{Lie}(\mathcal{P}) \subset \mathfrak{gl}_N(\mathbf{C}). \quad (59)$$

We note that the Lax matrix (44) transforms under the action of $q \in \mathcal{P}$ as

$$l(z) \mapsto l^q(z) = ql(z)q^{-1} \quad (60)$$

and this agrees with our general assumption (8).

This stage of Hamiltonian reduction is finite dimensional and, whatever parabolic subgroup \mathcal{P} and gauge fixing conditions are chosen, our formula (15) yields the desired r -matrix. To proceed further with this final Hamiltonian reduction we must now choose a concrete parabolic subgroup \mathcal{P} and impose the constraints $\mu^P(\alpha, \beta) = 0$ together with some gauge-fixing conditions. One could implement this simply using the Dirac reduction procedure once more to obtain finally an r -matrix for the elliptic spin Calogero–Moser system with the chosen quasi-parabolic structure. An alternative method however exists, and we will use that. This alternate route proceeds by constructing a gauge invariant extension of the Lax matrix. What is this *gauge invariant extension*? We have said that our Lax matrix transforms as (60). A gauge invariant extension of this is constructed by conjugating by a compensating gauge transformation that ‘undoes’ this gauge transformation. We imposed some gauge-fixing constraints $\chi^a = 0$ so as to choose (locally) one representative from each gauge orbit. Thus points in our total space can be described (locally) by coordinates (p, \tilde{q}) , where p is a point on the constraint surface $\chi^a = 0$ and \tilde{q} describes the gauge transformation along the orbit. The gauge invariant extension l^P of the Lax l is

$$l^P(z) = q^{-1}l(z)q$$

where if $l[p]$ transforms to $\tilde{q}^{-1}l\tilde{q}$ at the point (p, \tilde{q}) then $q(p, \tilde{q}) = \tilde{q}^{-1}$ so that $l^P(p, \tilde{q}) = l(p)$. Therefore, the twisted Lax matrix l^P is a gauge invariant extension. Now we have described in (7) how the r -matrix transforms under a gauge transformation, the desired r -matrix for the considered Hitchin system with the quasi-parabolic structure \mathcal{P} are found to be

$$r^P(z, z') = (q_1^{-1}q_2^{-1}r(z, z')q_1q_2 + q_1^{-1}q_2^{-1}\{l_2(z'), q_1\}q_2 + \frac{1}{2}[q_1^{-1}q_2^{-1}\{q_1, q_2\}, q_2^{-1}l_2(z')q_2])|_{\text{on-shell}}. \quad (61)$$

Observing that $q(p) = \text{Id}$, this expression reduces to (15), and so explains the origin of the remark we made in deriving our theorem. Thus by calculating (61) via the gauge invariant extension we will obtain the same r -matrix as determined by Dirac reduction. Reduction to the spin zero Calogero–Moser models yields the r -matrices of [35–37].

5. Example 2: the Feigin–Odesskii bracket via Poisson reduction

In a recent paper [38], the classical Sklyanin algebra has been derived by Hamiltonian reduction to the moduli space of complex structures on a principal GL_2 -bundle of degree one over an elliptic curve. In this section, we generalize the construction of [38] to the case of GL_N in order to obtain the classical Feigin–Odesskii algebra [8].

Let Σ_τ be an elliptic curve and let E_N be a principal GL_N -bundle over Σ_τ defined by the following gluing rules for a section s :

$$s(z + 1) = I_1 s(z) \quad s(z + \tau) = \Lambda(z) s(z). \tag{62}$$

Here

$$\Lambda(z) = I_2 \exp\left(-\frac{2\pi iz}{N}\right)$$

$$I_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & \varepsilon & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \varepsilon^{N-1} \end{pmatrix} \quad I_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and $\varepsilon = \exp\left(\frac{2\pi i}{N}\right)$.

The Čech cocycle condition determining this bundle follows from the commutation relation:

$$I_1 I_2 = \varepsilon^{-1} I_2 I_1.$$

It is easy to check that the holomorphic section of the corresponding determinant bundle $\det(E_N)$ is just a θ -function with a single simple zero on Σ_τ . We therefore conclude that $\deg(E_N) = 1$. Let us also note that

$$I_a = I_1^{a_1} I_2^{a_2} \quad a = (a_1, a_2) \in \mathbb{Z}_N \times \mathbb{Z}_N \tag{63}$$

forms a basis of gl_N .

In our construction of the Feigin–Odesskii bracket we use deformations of the complex structure on E_N that preserve the determinant bundle $\det(E_N)$ of E_N . In contrast to the previous section we will now use the Dolbeault picture to describe the complex structures on E_N . The operator

$$d_{\bar{A}} = k\bar{\partial} + \bar{A} : \mathcal{A}^{p,q}(\Sigma_\tau, E_N) \mapsto \mathcal{A}^{p,q+1}(\Sigma_\tau, E_N) \quad \text{Tr } \bar{A}(z, \bar{z}) = 0 \tag{64}$$

acts on the sections of E_N and defines a complex structure on E_N . Here $\bar{A} = \bar{A}(z, \bar{z}) d\bar{z}$ is a $(0, 1)$ -connection of E_N :

$$\bar{A}(z + 1) = I_1 \bar{A}(z) I_1^{-1} \quad \bar{A}(z + \tau) = I_2 \bar{A}(z) I_2^{-1}. \tag{65}$$

The constant k appearing in (64) may be further identified with a central charge. Note that the traceless condition $\text{Tr } \bar{A}(z, \bar{z}) = 0$ guarantees that the determinant of the deformed holomorphic bundle \tilde{E}_N , defined by $k\bar{\partial} + \bar{A}$ coincides with that of E_N .

Two complex structures \bar{A} and \bar{A}^f are called equivalent if they are related by the following transformation

$$\bar{A} \mapsto \bar{A}^f = f^{-1} \bar{A} f + f^{-1} k\bar{\partial} f \tag{66}$$

where $f = f(z, \bar{z})$ is a smooth SL_N -valued function on Σ_τ satisfying the following quasi-periodicity conditions:

$$f(z + 1) = I_1 f(z) I_1^{-1} \quad f(z + \tau) = I_2 f(z) I_2^{-1}. \tag{67}$$

Now we define the total Poisson space \mathbf{P} that will be reduced as a principal affinization over a cotangent bundle to the space of smooth sections of $\text{Aut}(E_N)$. A point on the respective base

is identified with a GL_N -valued field $g = g(z, \bar{z})$, satisfying the following quasi-periodicity conditions:

$$g(z+1) = I_1 g(z) I_1^{-1} \quad g(z+\tau) = I_2 g(z) I_2^{-1} \quad (68)$$

and a point on the fibre of \mathbf{P} is determined by a $(1, 0)$ -connection \bar{A} (65).

The Poisson brackets on \mathbf{P} are defined in the following way:

$$\begin{aligned} \{\bar{A}_{ij}(z, \bar{z}), \bar{A}_{kl}(w, \bar{w})\} &= (\bar{A}_{il}(z, \bar{z})\delta_{kj} - \bar{A}_{kj}(z, \bar{z})\delta_{il})\delta(z-w)\delta(\bar{z}-\bar{w}) \\ &+ k \left(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl} \right) \bar{\partial}\delta(z-w)\delta(\bar{z}-\bar{w}) \end{aligned} \quad (69)$$

$$\{g_{ij}(z, \bar{z}), \bar{A}_{kl}(w, \bar{w})\} = \left(g_{il}(z)\delta_{kj} - \frac{1}{N}g_{ij}(z)\delta_{kl} \right) \delta(z-w)\delta(\bar{z}-\bar{w})$$

$$\{g_{ij}(z, \bar{z}), g_{kl}(w, \bar{w})\} = 0$$

where k is a central charge. The space \mathbf{P} with this Poisson structure is a particular case of a general construction proposed by Polishchuk [9]. If we supplement the transformations of \bar{A} (66) with the simultaneous transformation of the field g

$$\bar{A} \mapsto \bar{A}^f = f^{-1}\bar{A}f + f^{-1}k\bar{\partial}f \quad g \mapsto g^f = f^{-1}gf \quad (70)$$

then equations (70) define a Poisson action on \mathbf{P} .

Let us consider the Poisson reduction with respect to the action (70). First we note that a generic field \bar{A} is in fact a pure gauge⁵

$$\bar{A} = f^{-1}k\bar{\partial}f \quad (71)$$

and we can choose $\bar{A} = 0$ as an appropriate gauge condition. Further, there are no residual gauge symmetries. To see this, consider a holomorphic function $f : \mathbb{C} \mapsto SL_N$ satisfying the quasi-periodicity conditions (67). By expanding in the basis (63) we find that

$$f_a(z+1) = \varepsilon^{-a_2} f_a(z) \quad f_a(z+\tau) = \varepsilon^{a_1} f_a(z).$$

Such a function must be a constant $f(z) = \text{Id}$. Thus the desired Poisson quotient is parametrized by a smooth section $L = L(z, \bar{z})$ of $\text{Aut}(E_N)$

$$L(z, \bar{z}) = f(z, \bar{z})g(z, \bar{z})f^{-1}(z, \bar{z}). \quad (72)$$

In order to calculate Poisson brackets on the reduced space we have to construct a gauge invariant functional $L = L[g, \bar{A}]$ which coincides with $L(z, \bar{z})$ on the surface of gauge fixing $\bar{A} = 0$. For this we solve equation (71) for $f(z, \bar{z})$ and substitute the solution into (72). Thus we get the desired gauge invariant extension

$$L[g, \bar{A}](z, \bar{z}) = f[\bar{A}](z, \bar{z})g(z, \bar{z})f^{-1}[\bar{A}](z, \bar{z}). \quad (73)$$

The calculation of the Poisson bracket between the entries of the matrix (72) is reduced to the calculation of an on-shell expression using the Poisson bracket (69) between the entries of (73). Note, that in performing these calculations we do not need to know the explicit dependence of $f_{ij}[\bar{A}]$ on \bar{A} . The only terms that enter the on-shell expression are

$$r_{ijkl}(z, \bar{z}; w, \bar{w}) = k \frac{\delta f_{ij}(z, \bar{z})}{\delta \bar{A}_{lk}(w, \bar{w})} \Big|_{\bar{A}=0} = -k \frac{\delta (f^{-1})_{ij}(z, \bar{z})}{\delta \bar{A}_{lk}(w, \bar{w})} \Big|_{\bar{A}=0}. \quad (74)$$

Due to equation (71) the function $r_{ijkl}(z, \bar{z}; w, \bar{w})$ turns out to be a Green's function for the $\bar{\partial}$ -operator

$$\bar{\partial}_z r_{ijkl}(z, \bar{z}; w, \bar{w}) = \left(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl} \right) \delta(z-w)\delta(\bar{z}-\bar{w}). \quad (75)$$

⁵ In other words this means that there are no moduli of (semi)stable complex structures on E_N .

Note that the function (74) can be found in the form $r(z, w) = r(z - w)$ and ‘unitarity’ in the sense that

$$r(z) = -r_{21}(-z). \tag{76}$$

In virtue of equations (65), (67), we have that $r(z)$ also possesses the following quasi-periodicity conditions:

$$r(z + 1) = (I_1 \otimes \text{Id})r(z)(I_1^{-1} \otimes \text{Id}) \quad r(z + \tau) = (I_2 \otimes \text{Id})r(z)(I_2^{-1} \otimes \text{Id}). \tag{77}$$

Thus $r_{ijkl}(z)$ defines a meromorphic function $r(z) : \mathbb{C} \mapsto sl_N \otimes sl_N$ with at most simple poles only at the points $z = n + m\tau, n, m \in \mathbb{Z}$. The residue t at the point $z = 0$ is the Killing form of the Lie algebra sl_N

$$t = \sum_{i,j} \left(e_{ij} \otimes e_{ji} - \frac{1}{N} e_{ii} \otimes e_{jj} \right). \tag{78}$$

Following the paper [15] there is a unique meromorphic function $r(z)$ satisfying the above conditions and, in particular, it is a solution of a classical Yang–Baxter equation

$$[r_{12}(z_1 - z_2), r_{23}(z_2 - z_3)] + [r_{12}(z_1 - z_2), r_{13}(z_1 - z_3)] + [r_{13}(z_1 - z_3), r_{23}(z_2 - z_3)] = 0. \tag{79}$$

Returning then to the calculation of the on-shell expression for the Poisson bracket

$$\{L_1[g, \bar{A}](z, \bar{z}), L_2[g, \bar{A}](w, \bar{w})\}|_{\bar{A}=0}$$

there are two types of contribution. The first type of contribution originates from the Poisson bracket between $g(z)$ and $f[\bar{A}](w)$, and the respective expression is

$$B_1 = 2k^{-1}L_1(z, \bar{z})L_2(w, \bar{w})r(z - w) - k^{-1}L_1(z, \bar{z})r(z - w)L_2(w, \bar{w}) - k^{-1}L_2(w, \bar{w})r(z - w)L_1(z, \bar{z}). \tag{80}$$

The second set of terms arise from the Poisson bracket between $f[\bar{A}](z)$ and $f[\bar{A}](w)$. These have the following form:

$$B_2 = -k^{-1}r(z - w)L_1(z, \bar{z})L_2(w, \bar{w}) + k^{-1}L_1(z, \bar{z})r(z - w)L_2(w, \bar{w}) + k^{-1}L_2(w, \bar{w})r(z - w)L_1(z, \bar{z}) - k^{-1}L_1(z, \bar{z})L_2(w, \bar{w})r(z - w). \tag{81}$$

Combining the expressions (80) and (81) we obtain the classical quadratic r -matrix

$$\{L_1(z, \bar{z}), L_2(w, \bar{w})\} = \frac{1}{k}[L(z, \bar{z}) \otimes L(w, \bar{w}), r(z - w)]. \tag{82}$$

The Jacobi identity for the Poisson bracket (82) follows from the classical Yang–Baxter equation (79).

In order to get an explicit expression for the r -matrix (74) we use the basis (63) and certain automorphic functions on \mathbb{C} satisfying the following quasi-periodicity properties:

$$\varphi_a(z + 1) = \varepsilon^{-a_2} \varphi_a(z) \quad \varphi_a(z + \tau) = \varepsilon^{a_1} \varphi_a(z). \tag{83}$$

It is easy to see that up to an overall factor for each $a \neq 0$ there exists a single function φ_a satisfying the above periodicity conditions (83) and having at most simple poles only at the points $n + m\tau, n, m \in \mathbb{Z}$. For $a = (0, 0)$ we simply set $\varphi_{(0,0)}(z) \equiv 1$. Then the elliptic r -matrix takes the following form:

$$r(z) = \frac{1}{N} \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N, a \neq 0} \varepsilon^{a_1 a_2} \varphi_a(z) I_a \times I_{-a}. \tag{84}$$

Let us return to the Poisson bracket (82). Consider the following set of functionals:

$$C_{ij} = \int_{\Sigma_\tau} d^2z \psi(z, \bar{z}) \bar{\partial} L_{ij}(z, \bar{z}) \quad i, j = 1, \dots, N \quad (85)$$

for all smooth functions ψ vanishing at the points $n + m\tau$, $n, m \in \mathbb{Z}$. We will prove that this is a set of weak Casimir functions⁶ for the Poisson bracket (82).

To this end let C be the matrix with entries (85). Straightforward calculation then shows that

$$\begin{aligned} \{C \otimes 1, L_2(w, \bar{w})\} &= \int_{\Sigma_\tau} d^2z \psi(z, \bar{z}) \bar{\partial} \{L_1(z, \bar{z}), L_2(w, \bar{w})\} \\ &= \int_{\Sigma_\tau} d^2z \psi(z, \bar{z}) \bar{\partial} \frac{1}{k} [L(z, \bar{z}) \otimes L(w, \bar{w}), r(z - w)]. \end{aligned} \quad (86)$$

Up to terms vanishing on the surface $C_{ij} = 0$ (for all i, j) expression (86) takes the form

$$\{C \otimes 1, L_2(w, \bar{w})\} = \int_{\Sigma_\tau} d^2z \psi(z, \bar{z}) \frac{1}{k} [L(z, \bar{z}) \otimes L(z, \bar{z}), t\delta(z - w)\delta(\bar{z} - \bar{w})] \quad (87)$$

where t is the residue of the r -matrix $r(z)$ at the point $z = 0$. Since t is adjoint invariant we see $\{C \otimes 1, L_2(w, \bar{w})\} = 0$. A consequence of this result is that the zero level surface for the functionals (85)

$$C_{ij} = 0 \quad (88)$$

is Poisson. That is, one can naturally define a Poisson bracket on the surface (88) such that the embedding of the surface into the initial space is a Poisson map.

Now the surface (88) is finite dimensional. In particular, the matrix $L(z, \bar{z})$ satisfying equations (88) can be represented as

$$L(z) = \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N} S_a \varphi_a(z) I_a \quad (89)$$

where S_a are c -numbers parametrizing the surface (88). Using some obvious properties of the automorphic functions φ_a one can easily show that relation (82) defines a quadratic Poisson bracket between the coordinates S_a . If we substitute solution (89) in equation (82) this is just the desired Feigin–Odesskii bracket, which is defined as a classical limit of the Feigin–Odesskii algebra [8].

Let us also note that all the Casimir functions of the Poisson brackets (82), (89) can be constructed with the help of our proposed reduction procedure. To see this we note the functional $\det g(z)$ yields a continuous set of Casimir functions for the Poisson brackets (69) of the unreduced space. Because the functional $\det g(z)$ is also gauge invariant it defines a continuous set of Casimir functions $\det L(z)$ for the Feigin–Odesskii brackets (82), (89). The desired algebraically independent Casimir functions for (82), (89) may then be defined as the following coefficients of the Laurent expansion for $\det L(z)$ around the marked point⁷ $z = 0$

$$C_0 = \oint_{\Gamma} \frac{\det L(z)}{z} \quad C_{-2} = \oint_{\Gamma} z \det L(z) \quad \dots \quad C_{-N} = \oint_{\Gamma} z^{N-1} \det L(z) \quad (90)$$

where Γ is a small contour around the point $z = 0$.

⁶ Functions C_A are said to form a set of weak Casimir functions if they generate a proper ideal in the Lie algebra of smooth functions with respect to the Poisson bracket.

⁷ It is easy to see that the residue of the function $\det L(z)$ at the point $z = 0$ is vanishing.

To conclude this section we would like to mention a relation between the matrix (89) and the Poisson brackets (82), and the higher dimensional elliptic top [39]. The equations of motion for this integrable system are usually given in the form

$$\frac{d}{dt}L(z) = \frac{1}{2}\{L(z), \mathcal{C}_{-2}\}_l \quad (91)$$

where $\{\cdot, \cdot\}_l$ stands for the following linear Poisson brackets

$$\{L_1(z), L_2(w)\}_l = \frac{1}{k}([L_1(z), r(z-w)] - [L_2(w), r_{21}(w-z)]) \quad (92)$$

and \mathcal{C}_{-2} is a quadratic Casimir function (90) of the Poisson brackets (82), (89). The same equations of motion (91) can however be written in an alternative manner with the aid of the quadratic Poisson brackets (82):

$$\frac{d}{dt}L(z) = \{L(z), H\}. \quad (93)$$

The Hamiltonian now has a very simple form

$$H = -S_{0,0}.$$

Note that the natural generalizations of the elliptic tops [39] associated with elliptic curves with multiple marked points can be obtained analogously in the framework of our construction. For this we require the functions $\psi(z, \bar{z})$, defining the weak Casimir functions (85) to vanish at each marked point.

6. Concluding remarks

In this paper, we have shown how the technique of Hamiltonian reduction enables us to calculate rather than guess some important ingredients in the theory of integrable systems. Although the technique is not always rigorous in the field theory context where conditionally convergent sums frequently arise, this approach does give us some important geometric information about the objects in question.

A stumbling block for the generalization of our results to curves of higher genus is the lack of a convenient basis both for meromorphic functions and for meromorphic differentials on a general Riemann surface. Thus in the paper [33], where a classical r -matrix is presented for Hitchin systems on an arbitrary Riemann surface of genus $g \geq 2$, it is mentioned that a formal expression for the r -matrix can be given in terms of a series in the Krichever–Novikov type basis [40, 41]. While we have shown that sums of the type (53) can be performed for the elliptic curve, for an arbitrary curve of genus $g \geq 2$ the series yields only a small amount of information about the geometric nature of the r -matrix in question.

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Appendix. Representations of the special functions $\phi(z, u, q)$ and $E(z, q)$

The meromorphic function $\phi(z, u)$ is defined by the following properties. First, it is automorphic with respect to the transformation $z \mapsto qz$, namely

$$\phi(q^{-1}z, u) = e^{2\pi i u} \phi(z, u). \quad (\text{A.1})$$

Second, on the unit circle $|z| = 1$ the function has one simple pole at the point $z = 1$ and the respective residue equals 1.

It is easy to check that the function with the above properties can be represented as

$$\phi(z, u) = \frac{1}{2\pi i} \frac{\theta_{11}(w+u)\theta'_{11}(0)}{\theta_{11}(w)\theta_{11}(u)} = \frac{1}{2\pi i} \frac{\sigma(w+u)}{\sigma(w)\sigma(u)} e^{-2\eta_1 u w} \quad (\text{A.2})$$

$$= \frac{1}{2\pi i} \left(\frac{1}{\omega} + \zeta(u) - 2\eta_1 u + \frac{\omega}{2} ((\zeta(u) - 2\eta_1 u)^2 - \wp(u)) + \dots \right). \quad (\text{A.3})$$

Here

$$\theta_{11}(w, \tau) = \sum_{m \in \mathbb{Z}} \exp(\pi i \tau (m + 1/2)^2 + 2\pi i (m + 1/2)(w + 1/2)) \quad (\text{A.4})$$

and $z = e^{2\pi i w}$. The function θ_{11} is the unique odd theta function, σ , ζ and \wp are the Weierstrass functions respectively of the same names, and $\eta_1 = \zeta(\frac{1}{2})$. Due to the automorphic properties of the θ -function (A.4) the right-hand side of equation (A.2) is in fact a function of z and $q = e^{2\pi i \tau}$.

On the other hand, using the above properties one can find series representations of the function (A.2) and thus prove that the properties uniquely define the desired meromorphic function. Due to the poles of the function (A.2) on each of the circles $|z| = |q|^m$, $m \in \mathbb{Z}$ the series representations depend on the choice of the annulus. In deriving the r -matrix we use two annuli, $1 < |z| < |q^{-1}|$ and $|q| < |z| < 1$. The series representations of (A.2) in these annuli are of the form

$$\phi(z, u) = \sum_{a \in \mathbb{Z}} \frac{z^a q^a e^{2\pi i u}}{q^a e^{2\pi i u} - 1} \quad 1 < |z| < |q^{-1}| \quad u \neq a + b\tau \quad a, b \in \mathbb{Z} \quad (\text{A.5})$$

and

$$\phi(z, u) = \sum_{a \in \mathbb{Z}} \frac{z^a}{q^a e^{2\pi i u} - 1} \quad |q| < |z| < 1 \quad u \neq a + b\tau \quad a, b \in \mathbb{Z}. \quad (\text{A.6})$$

The meromorphic function $E(z)$ is defined by analogous properties. It is automorphic with respect to the transformation $z \mapsto q^{-1}z$

$$E(q^{-1}z) = E(z) + 1. \quad (\text{A.7})$$

On the unit circle $|z| = 1$ the function has one simple pole at the point $z = 1$ and the respective residue equals 1. Finally, the function $E(z)$ has a vanishing moment in the annulus $1 < |z| < |q^{-1}|$, namely

$$\oint_{|z|=|q|^{-1/2}} \frac{dz}{2\pi i z} E(z) = 0. \quad (\text{A.8})$$

Using the θ -function (A.4) we may represent $E(z)$ as

$$E(z, q) = \frac{1}{2\pi i} \frac{\theta'_{11}(w, \tau)}{\theta_{11}(w, \tau)} - \frac{1}{2} = \frac{1}{2\pi i} (\zeta(w) - 2\eta_1 w) - \frac{1}{2}. \quad (\text{A.9})$$

The series representations for $E(z)$ in the annuli $1 < |z| < |q^{-1}|$ and $|q| < |z| < 1$ are of the form

$$E(z) = \sum_{a \in \mathbb{Z}, a \neq 0} \frac{z^a q^a}{q^a - 1} \quad 1 < |z| < |q^{-1}| \quad (\text{A.10})$$

and

$$E(z) = \sum_{a \in \mathbb{Z}, a \neq 0} \frac{z^a}{q^a - 1} - 1 \quad |q| < |z| < 1 \quad (\text{A.11})$$

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